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STATIONARY TRANSONIC SOLUTIONS OF A ONE—DIMENSIONAL HYDRODYNAMIC MODEL FOR SEMICONDUCTORS

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ABSTRACT. We study a hydrodynamic model for semiconductors where the energy equation is replaced by a pressure–density relationship. We construct artificial viscosity solutions, prove BV estimates independent of the viscosity coefficient and study the transonic weak limit. We also study the behavior of the limiting solution at the boundary for subsonic data. We find that a boundary layer can be formed on each side of the boundary and has a condition that determines the possible range of discontinuities for the density.

1. Introduction.

We are interested in the behavior of solutions for the hydrodynamic model for semiconductors introduced by Blotekjaer [B], which is capable of model hot electron effects which are not accounted for in the classical drift–diffusion model. A discussion about these models can be found in [M], [S] and [MRS].

A mathematical analysis for a simplified hydrodynamic model has been introduced by Degond and Markowich [DM1], [DM2]. Some preliminary results have been presented by Gardner, Jerome, and Rose [GJR], and also numerical simulations have been considered by Fatemi, Jerome and Osher [FJO].

In [MD1] they prove the existence of smooth solutions and a uniqueness result in the stationary subsonic one–dimensional model which is characterized by the smallness assumption on the current flowing through the device.

In [MD2] the authors prove existence and local uniqueness of smooth solutions of a three–dimensional steady state irrotational flow model based on the hydrodynamic equations, also under smallness assumptions on the data, which implies subsonic flow of electrons in the semiconductor device.

In this paper, we investigate the same simplified one–dimensional hydrodynamic model as in [MD1] in which the energy equation is replaced by the assumption that the pressure is a given function of the density only. We shall not make any assumption on the smallness of the boundary data.

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After appropriate scaling, the one-dimensional time-dependent system in the case of one carrier type (e.g. electrons) is

$$\rho_t + (\rho u)_x = 0 \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \rho\Phi_x = -\frac{\rho u}{\tau} \tag{1.2}$$

$$\Phi_{xx} = \rho - \mathcal{C}(x) \tag{1.3}$$

where $\rho(x,t)$, $u(x,t)$, $\Phi(x,t)$ denote the electron density, velocity and electrostatic potential respectively. $p = p(\rho)$ is the pressure-density relation which satisfies

(1.4) $\rho^2 p'(\rho)$ is strictly monotonically increasing from $(0,\infty)$ onto $(0,\infty)$.

The pressure function we use here is $p(\rho) = K\rho^\gamma$, where $\gamma > 1$ and $K > 0$.
Remark 1: What is actually needed of $p(\rho)$ is convexity and superlinear growth.
Remark 2: For $\gamma = 1$ it is necessary to request $-\Phi_x$ to be negative.
Here $\tau(u, \rho u)$ represents the momentum relaxation time which we assume

$$0 < \tau_0 \leq \tau(\rho, \rho u) \leq \tau_M \qquad \forall(\rho, \rho u) \in (0,\infty) \times \mathbb{R}. \tag{1.5}$$

The device domain is the x-interval $(0,1)$ and $\mathcal{C}(x) \in L^\infty(0,1)$ is the doping profile.
The system 1.1 to 1.3 has the boundary conditions

$$\rho(0,t) = \rho_0 \qquad \rho(1,t) = \rho_1 \tag{1.6}$$

$$\Phi(0,t) = 0 \qquad \Phi(1,t) = \Phi_1. \tag{1.7}$$

We consider here the steady state case $\rho_t = (\rho u)_t = 0$. Then, introducing the current density $j = \rho u$, the system (1.1)-(1.3) reduces to

$$j(x) = \text{const} \tag{1.8}$$

$$\left(\frac{j^2}{\rho} + \rho^\gamma\right)_x - \rho\Phi_x = -\frac{j}{\tau} \tag{1.9}$$

$$\Phi_{xx} = \rho - \mathcal{C}(x). \tag{1.10}$$

In [DM1] it is shown that for smooth solutions, j and Φ_1 are related by the current-voltage characteristic relationship

$$\Phi_1 = f(\rho_1, j) - f(\rho_0, j) + j\int_0^1 \frac{dx}{(\tau(\rho, j)\rho)(x)},$$

where $f(\rho, j) = \dfrac{j^2}{2\rho^2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}$ if $\gamma > 1$ or $f(\rho, j) = \dfrac{j^2}{2\rho^2} + \ln\rho$ if $\gamma = 1$.

In this case , for which we do not expect the solution to remain smooth , an appropriate boundary data for (1.8)-(1.10) then is

$$\rho(0) = \rho_0, \quad \rho(1) = \rho_1 \text{ and } \Phi(1) = \Phi_1. \tag{1.11}$$

We consider in this paper the "ε-viscosity" equations

$$j(x) = \text{const} > 0 \tag{1.12}$$

$$F(\rho)_x + S(w, \rho) + \varepsilon\rho_{xx} = 0 \tag{1.13}$$

$$w_x = \rho - \mathcal{C}(x) \tag{1.14}$$

where

$$F(\rho) = \frac{j^2}{\rho} + \rho^\gamma \text{ and } S(w,\rho) = -\rho w + \frac{j}{\tau(\rho, j)} \tag{1.15}$$

in the interval $I = (0,1)$ with boundary data

$$\rho(0) = \rho_0, \quad \rho(1) = \rho_1 \text{ and } w(0) = \omega_0. \tag{1.16}$$

If $j < 0$ we add negative "ε-viscosity".

We shall prove in the following sections that the limit of the ε-solution of (1.12)–(1.16) converges to a weak solution of the limiting problem as ε tends to zero. This solution (ρ, w) of (1.12)–(1.15) for $\varepsilon = 0$, will satisfy the classical "entropy condition" for ρ, namely ρ will be the sum of a Hölder–1/2 continuous function plus a monotone increasing function, which means the discontinuities of ρ can be only jumps from smaller to bigger values of ρ at the discontinuity points. These jumps can only occur in transonic regions to be described in Section 3. In case the current $j < 0$, we obtain that the density will be the sum of a Hölder–1/2 continuous function plus a monotone decreasing function.

Finally, we point out that a solution (ρ, w) of the limiting problem of (1.12)–(1.16) will give a solution (ρ, Φ) of (1.8)–(1.11) where $\Phi(x) = \int_0^{1-x} w(s)ds + \Phi_1$.

In section 2 we shall see that a solution, ρ, of the ε-equations (1.12)–(1.16) is bounded above by a constant and below by a positive constant depending only on the boundary data, the constant from (1.12), the exponent γ, and on condition (1.5). Also, we shall prove that $\varepsilon\rho_x$ is bounded uniformly in $\bar{I}$ independently of ε. Then, by the Leray–Schauder fixed point theorem, we see that the ε-viscosity problem has a unique solution in $C^{1,1}(\bar{I})$.

In section 3, we use the vanishing viscosity method to prove an existence theorem for a weak solution ρ of the problem (1.12)–(1.16) in the sense of the integral identity

$$\int_I (F(\rho)\varphi_x + S(w,\rho)\varphi)\,dx = 0 \tag{1.17}$$

which is valid for any $\varphi \in C_0^2(I) = \{\varphi \in C^2(I) : \operatorname{supp}(D^k\varphi) \subset I,\ k = 0,1,2\}$.

Also we show that, if $F(\rho_m) = \min_{\rho\in(0,\infty)} F(\rho)$, the function

$$\mathcal{H}(\rho) = (F(\rho) - F(\rho_m))\operatorname{sign}(\rho - \rho_m) \tag{1.18}$$

satisfies the condition that

$$\mathcal{H}(\rho)(x) + Cx \quad \text{is monotone increasing,} \tag{1.19}$$

where $C = \sup_I S(w, \rho^\varepsilon)$. That is $(\mathcal{H}(\rho))_x$ is a measure bounded below by $-C$.

The condition (1.19) represents the classical "entropy condition" for the transonic case in the sense of Olĭenik [O], Vol'pert [V], and Kružkov [K]. Note that with the problem being transonic you do not expect ρ but $\mathcal{H}(\rho)$ to be bounded variation.

Condition (1.19) will imply that weak solution ρ of (1.17) can be written as a sum of a Hölder–1/2 continuous function plus a monotone increasing function.

In order to obtain this convergence result we show that the function $\mathcal{H}(\rho^\varepsilon)(x)$ is of bounded variation in I with total variation norm denoted by $TV_I(\mathcal{H}(\rho^\varepsilon))$, bounded independently of ε.

Once we know the family $\{\mathcal{H}(\rho^\varepsilon)\}$ is uniformly of bounded variation, classical completeness and compactness theorems (Helly's theorem, Kolmogorov compactness condition, see [N]) assure us that we can extract a sequence $\{\mathcal{H}(\rho^{\varepsilon_n})\}$, $\varepsilon_n \to 0$ as $n \to \infty$, that converges pointwise in I and in every $L^p(I)$, $1 \le p < \infty$, to a bounded variation function $\mathcal{H}_0(x)$.

Further, $\{\mathcal{H}(\rho^{\epsilon_n})\}$ being uniformly bounded, $\phi(\mathcal{H}(\rho^{\epsilon_n}))$ converges to $\phi(\mathcal{H}_0)$ in $L^1(I)$, for any continuous function ϕ.

Thus, we shall define $\rho(x)$ as

$$\rho(x) = \lim_{n\to\infty} \mathcal{H}^{-1}(\mathcal{H}(\rho^{\epsilon_n})(x)) = \mathcal{H}^{-1}(\mathcal{H}_0)(x)$$

and we shall prove that ρ is an admissible solution to the problem (1.17)–(1.19).

Finally in Section 4 we discuss in which sense the boundary value is attained. We shall see that a boundary layer may be formed for the viscosity solution at both sides of the boundary. We prove these results for subsonic boundary data ρ_0 and ρ_1 (i.e. $\rho_0, \rho_1 > \rho_m$). On the right side of I (i.e. at $x = 1$) we obtain $\lim_{x\to 1^-} \rho(x) \le \rho_1$ and, if $\lim_{x\to 1^-} \rho(x) < \rho_1$ then $F(\rho_1) \le F\left(\lim_{x\to 1^-} \rho(x)\right)$ and $\lim_{x\to 1^-} \rho(x) < \rho_m$ (see Theorem 7). This means that, imposing Dirichlet boundary data on the downstream boundary, we get a condition on $\lim_{x\to 1^-} \rho(x)$ which says that either $\rho(x)$ achieves the prescribed value at $x = 1$ or a possible boundary layer can be formed which satisfies $\lim_{x\to 1^-} \rho(x)$ is a supersonic value lower than the supersonic conjugate value ρ_1^* of the subsonic prescribed value ρ_1 by F (i.e. $F(\rho_1) = F(\rho_1^*)$), as $F(\rho)$ is a strictly decreasing function of ρ for values $0 < \rho < \rho_m$.

On the left side of I (i.e. a the upstream boundary point $x = 0$) we prove that $\lim_{x\to 0^+} \rho(x) \ge \rho_m$ (see Theorem 8). This means that a possible boundary layer can be formed which remains in the subsonic region.

A Physical interpretation of these phenomena, as well as explicit examples are given by the author in [G].

The proofs of the theorems only make use of lemma 11, and it is expected that the same kind of results can be obtained for supersonic prescribed data. However, this analysis is not included here.

2. Existence of the viscosity solutions. In order to find bounds for the solution, ρ, of the ε–equation (1.13), we need bounds for the solution w of (1.14).

We shall soon see that an upper bound for the density ρ will depend on a lower bound for w. Thus, let $\widetilde{w}$ be the solution of

$$\widetilde{w}_x = -\mathcal{C}(x), \quad \text{with } \widetilde{w}(0) \le w_0. \tag{2.1}$$

Since $\mathcal{C}(x)$ is a bounded function then $\widetilde{w}$ is a Lipschitz function in $\overline{I}$. In our construction the densities ρ will always assumed to be non-negative. Let w be the solution of $w(0) = w_0 = \widetilde{w}(0)$ and $w_x = \rho - \mathcal{C}(x) \ge -\mathcal{C}(x)$, then

$$\widetilde{w} \le w \text{ in } I. \tag{2.2}$$

Similarly, in order to get a positive lower bound for ρ independent of ε, we shall need an upper bound for w. Thus, let B be an upper bound for all ρ^ϵ solutions of the ε–equation (1.13). We shall see below that B depends only on $\widetilde{w}, \tau_0, j$ and γ. Then, $w_x \le B - \mathcal{C}(x)$ in $|\overline{I}|$ and

$$w \le Bx + \widetilde{w} \le B + w_0 + \sup_I |\mathcal{C}(x)|,$$

uniformly in $\overline{I}$, independently of ε. That is, let w be the solution of $w(0) = w_0$ and $w_x = \rho - \mathcal{C}(x) \le B - \mathcal{C}(x)$, then

$$w \le B + w_0 + \sup_I |\mathcal{C}(x)|.$$

Boundedness of the density. We denote the ε-viscosity equation by

$$E_\varepsilon(\rho) = \left(\frac{j^2}{\rho} + \rho^\gamma\right)_x - \rho w + \frac{j}{\tau(\rho, j)} + \varepsilon \rho_{xx} = F(\rho)_x + S(\rho) + \varepsilon \rho_{xx}. \tag{2.4}$$

with j, w, γ prescribed, $\gamma > 1$ and we look at the problem

$$E_\varepsilon(\rho) = 0, \quad \rho(0) = \rho_0 \text{ and } \rho(1) = \rho_1. \tag{2.5}$$

E_ε is a second order quasilinear uniformly elliptic operator in I and ε is the ellipticity constant.

We see now that the following classical comparison principle applies to the E_ε operator.

Lemma 1. (Comparison principle). *is a strict supersolution of $E_\varepsilon(u) = 0$ (i.e., $E_\varepsilon(u_1) < 0$) and u_2 is a subsolution of $E_\varepsilon(u) = 0$ (i.e., $E_\varepsilon(u_2) \geq 0$) such that $u_2 \leq u_1$ in I, then $u_2 < u_1$ in I.*

Proof. Assume there exists an $x_0 \in I$ such that $u_2(x_0) = u_1(x_0)$, then $u_1 - u_2$ has a minimum at x_0. This means that $\nabla(u_1 - u_2)(x_0) = 0$ and $(u_1 - u_2)_{xx}(x_0) \geq 0$. Then

$$\begin{aligned} E_\varepsilon(u_1)(x_0) - E_\varepsilon(u_2)(x_0) &= (F'(u_1(x_0))\nabla u_1(x_0) - F'(u_2(x_0))\nabla u_2(x_0) \\ &\quad - [(u_1 - u_2)w](x_0) + \frac{j}{\tau(u_1(x_0))} - \frac{j}{\tau(u_2(x_0))} \\ &\quad + \varepsilon(u_1 - u_2)_{xx}(x_0) \\ &= \varepsilon(u_1 - u_2)_{xx}(x_0) \geq 0 \end{aligned}$$

where $F'(u) = -\frac{j^2}{u^2} + \gamma u^{\gamma - 1}$. Since $E_\varepsilon(u_2) \geq 0$, then it follows that at x_0, $E_\varepsilon(u_1)(x_0) \geq 0$ which contradicts that u_1 is a strict supersolution. □

The next lemma is a consequence of the previous comparison principle. It will allow us to compare any subsolution of the E_ε operator with a continuous family of supersolutions provided that all supersolutions remain larger than the subsolutions on the boundary and that there is at least one supersolution in the family that strictly controls such subsolution. This is a classical result widely used with elliptic operators satisfying a comparison principle as the one in Lemma 2. We shall write the guidelines of the proof.

Lemma 2. *Let E_ε be an operator that admits a comparison principle as lemma 1. If u_1 is a subsolution of $E_\varepsilon(v) = 0$ in I and u^t is a continuous family of supersolutions of $E_\varepsilon(v) = 0$ in I, for $t \in [0, T]$ such that $u^0 > u_1$ and $u^t|_{\partial I} > u_1|_{\partial I}$ for all $t \in [0, T]$, then $u^t > u_1$ in I for all $t \in [0, T]$.*

Proof. Consider $A_\theta = \{t \in [0, \theta] : u^t > u_1 \text{ in } I\}$, that is θ the smallest positive value such that $u^t > u_1$. A_θ is not empty since $t = 0$ is in A_θ.

We shall see that A_θ is open and closed in $[0, T]$, then $A_\theta = [0, T]$, which completes the proof.

In order to show that A_θ is open, we let $t \in A_\theta$, then $u^t - u_1$ is a positive continuous function in $\bar{I}$. Let $\delta = \min(u^t - u_1) > 0$, then, since the family u^t is continuous, there exist a μ_0 such that $|u^{t+\mu} - u^t| < \delta$ for all $|\mu| \leq \mu_0$. Therefore, $u^{t+\mu} > u_1$ for all $|\mu| \leq \mu_0$.

Finally, we prove that A_θ is closed by showing that $\theta \in A_\theta$. Let $t^n \to \theta$, $t^n < \theta$, since the family of u^t is continuous in t, then, $u^{t^n} \to u^\theta$, $u^\theta(x) \geq u_1(x)$, $x \in I$, and $u^\theta|_{\partial I} > u_1|_{\partial I}$. Then, by Lemma 1, $u^\theta > u_1$ in $\bar{I}$, and therefore $\theta \in A_\theta$.

Note: Lemmas 1 and 2 apply to a general equation $F(D^2 u, Du, u, x) = 0$, monotone in $D^2 u$.

Similar lemmas are obtained changing subsolutions by supersolution and vice versa, and also by changing the direction of the inequalities.

Next, we find a supersolution $\overline{\rho}$ for the problem (2.5) that is independent of ε and depends on w only through its minimum. For this, we write the operator E_ε, defined in (2.4), in the following form

$$\left(-\frac{j^2}{\rho^2}+\gamma\rho^{\gamma-1}\right)\rho_x-\rho w+\frac{j}{\tau}+\varepsilon\rho_{xx}=F'(\rho)\rho_x+S(\rho)+\varepsilon\rho_{xx}. \tag{2.6}$$

As before, we think on γ, j and w prescribed and $w \geq \mu_0$ a constant (later we will use $\mu_0 = \inf_I \tilde{w}$ independent of ε). We try a linear function $\overline{\rho} = Ax + M$, $\overline{\rho} > 0$ in I. Then

$$E_\varepsilon(\overline{\rho})=\left(-\frac{j^2}{(Ax+M)^2}+\gamma(Ax+M)^{\gamma-1}\right)A-(Ax+M)w+\frac{j}{\tau(\overline{\rho})}.$$

Then, using that $\tau(\overline{\rho}) \geq \tau_0$ and $w \geq \mu_0$ where τ_0 and μ_0 are independent of ε and ρ, we get

$$E_\varepsilon(\overline{\rho})\leq\left(-\frac{j^2}{(Ax+M)^2}+\gamma(Ax+M)^{\gamma-1}\right)A-(Ax+M)\mu_0+\frac{j}{\tau_0}.$$

Therefore, taking $A < 0$ the right-hand side of the above inequality is bounded by

$$\left(-\frac{j^2}{(A+M)^2}+\gamma M^{\gamma-1}\right)A-(A+M)\mu_0+\frac{j}{\tau_0}.$$

Since $\gamma - 1 > 0$, we put $A = -\frac{M}{2}$, so that $\overline{\rho} > 0$, and M is then taken large enough so that this expression becomes negative. Also, since $\overline{\rho}(0) = M$ and $\overline{\rho}(1) = \frac{M}{2}$ we impose $\frac{M}{2} > \max\{\rho_0, \rho_1\}$. Therefore, there exists a B independent of ε, such that for $M > B$

$$E_\varepsilon(\overline{\rho}) < 0, \quad \text{with } \overline{\rho} = -\frac{Mx}{2} + M \tag{2.7}$$

and $\overline{\rho}|_{\partial I} > \rho|_{\partial I}$ for every ε.

We now apply lemma 2 to the continuous family of $\overline{\rho}_M = -\frac{M}{2}x + M$, satisfying (2.7), as M moves. Indeed, since each ρ^ε solution of $E_\varepsilon(\rho^\varepsilon) = 0$ is bounded by a constant C_ε, then taking $B_\varepsilon = 2C_\varepsilon$, we get that $\rho^\varepsilon < \overline{\rho}_{B_\varepsilon}$, and that $\overline{\rho}_{B_\varepsilon}$ is a supersolution if $B_\varepsilon \geq B$, where B is the fixed constant from (2.7). (If $B_\varepsilon < B$, then $\rho_\varepsilon < \overline{\rho}$ and there is nothing to prove.)

Thus, our u^t family defined

$$u^t = \overline{\rho}_{B_\varepsilon - t} \quad 0 \leq t \leq B_\varepsilon - B, \tag{2.8}$$

satisfies that u^t are all supersolutions of $E_\varepsilon(u) = 0$ and also is a continuous family in the sense that $u^t \to u^{t_0}$ in $C^0([0,1])$ whenever $t_n \to t_0$ in $[0, B_\varepsilon - B]$.

Then, if ρ^ε is a solution of (2.5) then $\rho^\varepsilon < u^0 = \overline{\rho}_{B_\varepsilon}$ and $\rho^\varepsilon|_{\partial I} \leq u^t|_{\partial I}$ for all $t \in [0, B_\varepsilon - B]$. Then by lemma 2, $\rho^\varepsilon < u^t$, $t \in [0, B_\varepsilon - B]$. Particularly

$$\rho^\varepsilon < u^{B_\varepsilon - B} = \overline{\rho}. \tag{2.9}$$

Hence, we have obtained that

$$\rho^\varepsilon < B \text{ in } \overline{I}, \tag{2.10}$$

where B depends only on $w_0, \tau_0, \gamma, j, \rho_0$ and ρ_1 and is independent of ε.

Next we shall find a subsolution $\underline{\rho}$ of the problem (2.5) which is independent of ε. By analogous lemmas to Lemmas 1 and 2 we shall obtain a lower bound for ρ^ε solution of problem (2.5) independent of ε. Again we assume here w prescribed and $w \le \mu_1$.

Indeed, we set $\underline{\rho} = K$ constant. Since we want

$$E_\varepsilon(\underline{\rho}) > I \text{ in } 0, \text{ with } \underline{\rho}|_{\partial I} \le \rho^\varepsilon|_I, \tag{2.11}$$

and using (1.5), (2.3), K needs to satisfy

$$E_\varepsilon(K) = -Kw + \frac{j}{\tau(K)} \ge -K\mu_1 + \frac{j}{\tau_M} > 0.$$

This implies

$$K < \frac{j}{\tau_M \mu_1}.$$

From the boundary conditions of (2.10), we need K to be smaller than the boundary data. Hence we take

$$\underline{\rho} < \min\left\{\rho_0, \rho_1, \frac{j}{\tau_M \mu_1}\right\},$$

and

$$\underline{\rho} < \rho^\varepsilon$$

holds for every ε and ρ independent of ε.

This concludes the proof of the following lemma.

Lemma 3 (Uniform boundedness of the ε-viscosity solutions). *If ρ^ε is a solution of the problem $E_\varepsilon(\rho^\varepsilon) = 0$ in $I = (0,1)$, with $\rho^\varepsilon(0) = \rho_0$ and $\rho^\varepsilon(1) = \rho_1$, where E_ε is defined in (2.4), then, if $w \ge \mu_0$*

$$\sup_I \rho^\varepsilon \le B \text{ for all } \varepsilon \tag{2.11}$$

where B is a constant depending only on μ_0, τ_0, γ, j ρ_0 and ρ_1, and if $w \le \mu_1$

$$\inf_I \rho^\varepsilon \ge \min\left\{\rho_0, \rho_1, \frac{j}{\tau_M \mu_1}\right\} \text{ for all } \varepsilon. \tag{2.12}$$

Remark. The bounds do not depend on the behavior of F near zero, only on the behavior of F for say $\rho > \min\{\rho_0, \rho_1\}$.

Next, we shall see that $\varepsilon\rho^\varepsilon_x$ is bounded uniformly in I, independent of ε, where ρ^ε is the solution of problem (2.5).

Lemma 4. *If ρ^ε is a solution of the problem $E_\varepsilon(\rho^\varepsilon) = 0$ in $I = (0,1)$, and let C be a constant such that $\frac{1}{C} \le \rho_\varepsilon$ and $\|\rho_\varepsilon\|_{L^\infty}, \|w\|_{L^\infty} \le C$ then $\varepsilon\rho^\varepsilon_x$ is uniformly bounded in $\overline{I}$ by $\lambda(C)$.*

Proof. Set $\sigma(x) = \rho^\varepsilon(\varepsilon x)$. Then $\sigma_x = \varepsilon\rho^\varepsilon_x$ and $\sigma_{xx} = \varepsilon^2\rho^\varepsilon_{xx}$, and replacing this in (2.6) we get

$$F'(\sigma)\frac{\sigma_x}{\varepsilon} + S(\sigma) = -\frac{\varepsilon}{\varepsilon^2}\sigma_{xx}.$$

In particular

$$|\sigma_{xx}| \le D(|\sigma_x| + 1) \text{ in } I' = (0, \varepsilon^{-1}), \text{ for } D = D(C). \tag{2.13}$$

By (2.11) $D \le C$ independent of ε. Letting $x_0 \in I$, we construct the following differentiable barrier functions

$$b^+(x - x_0) = -\frac{C}{\delta^2}((x - x_0) - \delta)^2 + (C + \sigma(x_0)) \tag{2.14}$$

defined in the interval $[x_0, x_0 + \delta]$. We assume $x_0 + \delta \in I'$, if not we do a similar construction in $[x_0 - \delta, x_0]$.

$b^+(r)$ is a section of a parabola with vertex at $C + \sigma(x_0)$ and axis at $r = \delta$ and second derivative $\frac{-2C}{\delta^2}$, so that $b(0) = \sigma(x_0)$. Also, define

$$b^-(x - x_0) = -b^+(x - x_0) \text{ in } [x_0, x_0 + \delta]. \tag{2.15}$$

If $x_0 + \delta \notin I'$ then $x_0 + \delta \ge \varepsilon^{-1}$, then we do a symmetric construction in $[x_0 - \delta, x_0]$ (as we mentioned above): $b^+(r)$ is a section of a parabola with vertex at $C + \sigma(x_0)$ and axis $r = \delta$ and second derivative $\frac{-2C}{\delta^2}$, so that $b(0) = \sigma(x_0)$, that is $b^+(x_0 - x)$ for $x \in [x_0 - \delta, x_0]$, and $b^-(x_0 - x) = -b^+(x_0 - x)$ also in $[x_0 - \delta, x_0]$.

We check that b^+ and b^-, defined in (2.14) and (2.15), are supersolutions and subsolution of inequality (2.13) in $[x_0, x_0 + \delta]$, respectively. Indeed

$$b^+_{xx} = \frac{-C}{\delta^2} \qquad \text{and} \qquad |b^+_x| \le \frac{4C}{\delta} \quad \text{for } |x - x_0| < \delta$$

. Hence, $b^+_{xx} < -D(|b^+_x| + 1)$, if δ is chosen small enough.

Then, b^+ is a strict supersolution of inequality (2.13) in the interval $[x_0, x_0 + \delta]$ where $b^+(x_0) = \sigma(x_0)$ and $b^+(x_0 + \delta) = C + \sigma(x_0) > \sigma(x_0 + \delta)$.

Analogously, b^- is a strict subsolution in the same interval $[x_0, x_0 + \delta]$. Then by the standard maximum principle applied to the inequality 2.13,

$$b^- \le \sigma \le b^+ \quad \text{in } [x_0, x_0 + \delta]$$

where δ is independent of ε and x_0.

Assuming σ being differentiable, approaching the first derivatives at the point x_0 from the right we get

$$-\frac{2C}{\delta} = b^-_x(x_0) \le \sigma_x(x_0) \le b^+_x(x_0) = \frac{2C}{\delta}.$$

Hence $|\sigma_x(x_0)| \le \frac{2C}{\delta}$ where C and δ are independent of ε and $x_0 \in \overline{I}'$. Thus we have obtained that

$$|\varepsilon\rho^\varepsilon_x| \le \frac{2C}{\delta}, \text{ independent of } \varepsilon \text{ and uniformly in } \overline{I}. \tag{2.16}$$

We finish this section by showing that problem (2.5) is solvable. We use a special case of the Leray-Schauder fixed point theorem. A proof for this theorem can be found in [GT] Sc. 10.2.

Theorem. *Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself and suppose there exists a constant M such that*

(2.17) $$\|x\|_B \le M$$

for all $x \in \mathcal{B}$ and $\sigma \in [0,1]$ satisfying $x = \sigma T x$, then T has a fixed point.

In order to apply this theorem, we construct the operator $T_\delta(v) : C^{0,1}(I) \mapsto C^{0,1}(I)$, for $\delta << \rho_0, \rho_1$, the following way:
Given $v \in C^{0,1}(I)$, we solve

(2.18) $$w_x = v^+ - C(x), \quad w(0) = w_0$$

and

(2.19) $$\varepsilon\rho_{xx} = -F^\delta(v^+)_x + S(w, v^+) \text{ in } I, \quad \rho(0) = \rho_0, \rho(1) = \rho_1,$$

where

$$F^\delta(v) = \begin{cases} F(v) & \text{for } v > \delta \\ F(\delta) & \text{for } v \le \delta, \end{cases}$$

and we define $\rho = T_\delta(v)$. Then the following theorem holds.

Theorem 1. *T_δ, defined as above, has a fixed point.*

Proof. T_δ is compact since, from equation (2.19),

(2.20) $$\|\rho\|_{C^{1,1}(\bar{I})} \le K(\varepsilon, \delta, M), \qquad \text{for } \|v\|_{C^{0,1}(\bar{I})} \le M.$$

Now if $\tilde{\rho} = \sigma T_\delta(\tilde{\rho})$, i.e.

$$\tilde{\rho}_{xx} = -\frac{\sigma}{\varepsilon}\left[F^\delta(\tilde{\rho}^+)_x + S(w, \tilde{\rho}^+)\right], \quad \tilde{\rho}(0) = \sigma\rho_0 \text{ and } \tilde{\rho}(1) = \sigma\rho_1,$$

it follows from Lemma 3 and Lemma 4 that

$$0 \le \tilde{\rho} \le B$$

and

$$|\tilde{\rho}_x| \le K\left(\frac{\sigma}{\varepsilon}, \delta\right),$$

then

(2.21) $$\|\tilde{\rho}\|_{C^{0,1}(I)} \le M,$$

so that (2.17) is satisfied.

By estimate (2.20), T_δ maps bounded sets of $C^{0,1}(\bar{I})$ into bounded sets of $C^{1,1}(\bar{I})$, which are precompact (by Arzela–Ascoli's theorem) in $C^{0,1}(I)$.

In order to show the continuity of T_δ, we let v_m, $m = 1, 2, \ldots$, converge to v in $C^{0,1}(\bar{I})$. Then, since the sequence $\{T_\delta v_m\}$ is precompact in $C^{0,1}(\bar{I})$, every subsequence has a convergent subsequence. Let $\{T_\delta \bar{v}_m\}$ be such a convergent subsequence with limit $\rho \in C^{0,1}(\bar{I}) \cap C^{1,1}(\bar{I})$, then, since

$$\varepsilon\rho_{xx} + (F^\delta(v^+))_x + (S(w, v^+) = \lim_{m \to \infty}\{\varepsilon T_\delta \bar{v}_m + F^\delta(\bar{v}_m^+)_x + S(w, \bar{v}_m^+)\} = 0,$$

we must have $\rho = T_\delta v$, and hence the sequence $\{T_\delta v_m\}$ converges to ρ. Hence we apply the Leray–Schauder theorem and we get that T_δ has a fixed point ρ, that is, from (2.18) and (2.19)

$$w_x = \rho^+ - \mathcal{C}(x)$$

and

$$E_\varepsilon^\delta(\rho) = \varepsilon\rho_{xx} + F^\delta(\rho^+)_x + S(w, \rho^+) = 0$$

in I, $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$. It follows that taking, from (2.2) and (2.3), $\mu_0 = \inf_I w$ and $\mu_1 = B + w_0 + \sup_I|\mathcal{C}(x)|$, respectively, and applying Lemma 3 and the remark after it, we obtain

$$\min\left\{\rho_0, \rho_1, \frac{j}{\tau_M(B + w_0 + \sup_I|\mathcal{C}(x)|)}\right\} \le \rho \le B \tag{2.24}$$

where B depends on μ_0, τ_0, γ, j, ρ_0 and ρ_1.

In particular, $F^\delta(\rho^+) = F(\rho)$ for any δ less than the left side of (2.24), then replacing in (2.22) and (2.23), we obtain that ρ solves

$$w_x = \rho - \mathcal{C}(x)$$

and

$$E_\varepsilon(\rho) = \varepsilon\rho_{xx} + F(\rho)_x + S(w, \rho) = 0$$

in I, $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$ and $w(0) = w_0$, which is the original system.

3. Existence of a weak solution of the limiting problem and entropy condition. In order to obtain a convergence result for the ε–viscosity solutions, we need to prove the function $\mathcal{H}(\rho^\varepsilon)(x)$ defined in (1.17) of bounded variation, with a TV–norm bounded independently of ε: that is the function $\mathcal{H}(\rho^\varepsilon) = (F(\rho^\varepsilon) - F(\rho_m))\,\mathrm{sign}(\rho^\varepsilon - \rho_m)$ satisfies $TV_I(\mathcal{H}(\rho^\varepsilon)) \le K$, K independent of ε.

Indeed, the choice of $\mathcal{H}$ comes from multiplying the equation (2.4) by a smooth approximation of $\mathrm{sign}(\rho - \rho_m)$, where ρ_m satisfies

$$F(\rho_m) = \min_{\rho\in(0,\infty)} F(\rho) = \min_{\rho\in(0,\infty)}\left(\frac{j}{\rho} + \rho^\gamma\right).$$

We denote $H_\delta(\rho) = \mathrm{sign}_\delta(\rho - \rho_m)$ the regularization of $\mathrm{sign}(\rho - \rho_m)$. Multiplying the equation (2.4) by $H_\delta(\rho^\varepsilon)$, we obtain

$$H_\delta(\rho^\varepsilon)F'(\rho^\varepsilon)\rho^\varepsilon_x + H_\delta(\rho^\varepsilon)S(\rho^\varepsilon) = -\varepsilon\rho^\varepsilon_{xx}H_\delta(\rho^\varepsilon), \tag{3.1}$$

where we recall

$$F'(\rho) = -\frac{j^2}{\rho^2} + \gamma\rho^{\gamma-1} \quad \text{and } S(\rho) = -\rho w + \frac{j}{\tau(\rho, j)}. \tag{3.2}$$

Now, let

(3.3) $$\mathcal{H}_\delta(\rho) = (F(\rho) - F(\rho_m))H_\delta(\rho - \rho_m).$$

Since F is a smooth function of ρ and has a minimum at ρ_m then $|F(\rho) - F(\rho_m)| \leq F''(\rho_m)|\rho - \rho_m|^2 + \mathcal{O}((\rho - \rho_m)^3)$. Also $(\text{sign}_\delta(\rho - \rho_m))'$ is zero for $|\rho - \rho_m| > \delta$ and bounded by δ^{-1} for $|\rho - \rho_m| < \delta$. Then

$$|(\text{sign}_\delta(\rho - \rho_m))'(F(\rho) - F(\rho_m))| \leq F''(\rho_m)\delta + \mathcal{O}(\delta^2) = \mathcal{O}(\delta).$$

Therefore, if ρ^ε is a $C^{1,1}(I)$ function (by theorem 1) then $(H_\delta F')(\rho^\varepsilon)(x)$ is continuous and ρ^ε is absolutely continuous then (see Natanson [N]) we get the function defined in (3.3) satisfies

(3.4) $$\mathcal{H}_\delta(\rho^\varepsilon)(x) = \int_0^x \mathcal{H}'_\delta(\rho^\varepsilon(x))d(\rho^\varepsilon(x)) = \int_0^x [(H_\delta F')(\rho^\varepsilon) + \mathcal{O}(\delta)]\rho^\varepsilon_x dx = \int_0^x (\mathcal{H}_\delta(\rho))_x dx$$

where the $\mathcal{O}(\delta)$ do not depend on ρ, and $\mathcal{O}(\delta) \to 0$ as $\delta \to 0$.

Combining (3.4) in (3.1) we obtain

(3.5) $$(\mathcal{H}_\delta(\rho^\varepsilon))_x - \mathcal{O}(\delta)\rho^\varepsilon_x + G_\delta(\rho^\varepsilon) = -\varepsilon(P_\delta(\rho^\varepsilon))_{xx} + \varepsilon H'_\delta(\rho^\varepsilon)(\rho^\varepsilon_x)^2$$

with

(3.6) $$G_\delta = H_\delta S \quad \text{and} \quad P_\delta = \int H_\delta.$$

We would like to prove now that $\mathcal{H}_\delta(\rho^\varepsilon)(x)$ is a function of bounded variation with total variation norm independent of δ and ε.

From classical theory on Lebesgue integral (see Natanson, [N] 4.IX), since the function $((H_\delta F')(\rho^\varepsilon) + \mathcal{O}(\delta))\rho^\varepsilon_x$ is integrable in I, then

(3.7) $$TV_I(\mathcal{H}_\delta(\rho^\varepsilon)) = \int_0^1 |(\mathcal{H}_\delta(\rho^\varepsilon))_x| dx$$

where $TV_I(\mathcal{H}_\delta(\rho^\varepsilon))$ denotes the total variation norm of the function $\mathcal{H}_\delta(\rho^\varepsilon)(x)$ on I.

Previously, we prove the following lemma.

Lemma 5. *For each fixed ε, there exists a $\delta_0 = \delta_0(\varepsilon)$ such that the last term of (3.5), satisfies*

(3.8) $$\varepsilon \int_0^1 H'_\delta(\rho^\varepsilon)(\rho^\varepsilon_x)^2 dx \leq K$$

where K is a constant independent of ε and δ, for every $\delta < \delta_0$.

Proof. We integrate in I the equation (3.5) and obtain

(3.9) $$\mathcal{H}_\delta(\rho^\varepsilon(0)) - \mathcal{H}_\delta(\rho^\varepsilon(1)) + \int_0^1 G_\delta(\rho^\varepsilon)dx - \int_0^1 \mathcal{O}(\delta)\rho^\varepsilon_x = = -\varepsilon(P_\delta(\rho^\varepsilon))_x|_0^1 + \varepsilon \int_0^1 H'_\varepsilon(\rho^\varepsilon)(\rho^\varepsilon_x)^2 dx.$$

By lemma 3 in Section 2, we have that

$$G_\delta(\rho^\varepsilon) = H_\delta(\rho^\varepsilon)S(\rho^\varepsilon) \leq S(\rho^\varepsilon) \leq K$$

where K depends on B, w_0, ρ_0, and ρ_1 and independent of ε and δ.

By lemma 4 we know that $\varepsilon\rho^\varepsilon_x \leq C$, C independent of ε, then $\mathcal{O}(\delta)\rho^\varepsilon_x \leq C$ for $\delta < \delta_0(\varepsilon)$.

The terms $\mathcal{H}_\delta(\rho^\varepsilon(0))$ and $\mathcal{H}_\delta(\rho^\varepsilon(1))$ are also bounded independently of ε and δ, since $\rho^\varepsilon(0) = \rho_0$ and $\rho^\varepsilon(1) = \rho_1$ for all ε and F is a continuous function of ρ.

Finally, it remains to analyze the term $\varepsilon(P_\delta(\rho^\varepsilon))_x$ evaluated at the points 0 and 1. Using lemma 4 once again, and (3.6), we get

$$\varepsilon(P_\delta(\rho^\varepsilon))_x|_0^1 = \varepsilon H_\delta(\rho)\rho_x|_0^1 \le \varepsilon\rho_x|_0^1 \le C$$

where C is independent of δ and ε.

Thus from equation (3.9) we obtain that

$$\varepsilon \int_0^1 H'_\delta(\rho^\varepsilon)(\rho_x^\varepsilon)^2 dx \le K, \quad \delta < \delta_0(\varepsilon)$$

where K is a constant independent of ε and δ, which comes out from the sum of the upper bounds of the remaining terms of the equation (3.9).

Now we are in conditions to prove that the total variation of $\mathcal{H}_\delta(\rho^\varepsilon)$ is bounded independently of δ and ε.

Lemma 6. *For each fixed ε, there exists a $\delta_0 = \delta_0(\varepsilon)$ such that*

$$TV_I(\mathcal{H}_\delta(\rho^\varepsilon)) \le K \tag{3.10}$$

for $\delta < \delta_0$ and K a constant independent of δ and ε where the total variation norm has been defined in (3.7).

Proof. Since $|\rho_{xx}| \le C(\varepsilon,\delta)$, $\mathcal{H}_\delta(\rho^\varepsilon)$ is of bounded total variation in I. Hence, in order to estimate $TV_I(\mathcal{H}_\delta(\rho^\varepsilon))$ it is enough to estimate

$$\int |(\mathcal{H}_\delta(\rho^\varepsilon))_x|$$

in any number of intervals I_n, $0 \le n \le k$ where ρ_x^ε remains positive or negative. Let I_n be such a family, then we take $I_n = (a_n, b_n)$ where $\rho_x^\varepsilon = 0$ both at a_n and b_n; or $a_0 = 0$ or $b_k = 1$.

We compute

$$(F_{\delta,\varepsilon})_k = \sum_{0\le n\le k} \int_{I_n} |(\mathcal{H}_\delta(\rho^\varepsilon))_x|\, dx. \tag{3.11}$$

Since ρ_x^ε does not change sign in I_n, and $\mathcal{H}_\delta(\rho^\varepsilon)$ is a monotone function of ρ^ε then $\mathcal{H}'_\delta(\rho^\varepsilon)$ does not change sign in I_n either. The integral then commutes with the absolute value, and (3.11) becomes

$$(F_{\delta,\varepsilon})_k = \sum_{0\le n\le k} \left|\int_{I_n} (\mathcal{H}_\delta(\rho^\varepsilon))_x dx\right|. \tag{3.12}$$

Integrating equation (3.5) on I_n, F_k becomes bounded by

$$\begin{aligned}(F_{\delta,\varepsilon})_k \le& \sum_{0\le n\le k}\left[|\mathcal{O}(\delta)\rho_x^\varepsilon| + \sup_I |G_\delta(\rho^\varepsilon(x))|\right]|I_n| \\ &+ \varepsilon\sum_{0\le n\le k}\left|\int_{I_n}(P_\delta(\rho^\varepsilon))_{xx}dx\right| + \varepsilon\sum_{0\le n\le k}\left|\int_{I_n} H'_\delta(\rho^\varepsilon)(\rho_x^\varepsilon)^2dx\right|.\end{aligned} \tag{3.13}$$

Since $H'_\delta(\rho^\varepsilon)(\rho_x^\varepsilon)^2$ is non-negative in I, then by lemma 5, for each fixed ε, the last term can be bounded as

$$\varepsilon\int_{\bigcup_{n=0}^k} H'_\delta(\rho^\varepsilon)(\rho_x^\varepsilon)^2dx \le \varepsilon\int_0^1 H'_\delta(\rho^\varepsilon)(\rho_x^\varepsilon)^2dx \le K$$

for $\delta < \delta_0(\varepsilon)$, with K independent of ε and δ.

The first term in the right hand side of (3.13) is estimated by similar estimates as the ones obtained for these terms in lemma 5, so that it is bounded by

$$\sup_I(|\mathcal{O}(\delta)\rho_x^\varepsilon| + |G_\delta(\rho^\varepsilon(x))| \sum_{0\le n\le k} |I_n| < K$$

for $\delta < \delta_0(\varepsilon)$ and K independent of ε and δ.

Finally it remains the terms that contains $\varepsilon\left|\int_{I_n}(P_\delta(\rho^\varepsilon))_{xx}dx\right|$. Since by (3.6)

$$\varepsilon\int_{I_n}(P_\delta(\rho^\varepsilon))_{xx}dx = (\varepsilon P_\varepsilon(\rho^\varepsilon))_x\Big|_{a_n}^{b_n} = \varepsilon H_\delta(\rho^\varepsilon)\rho_x^\varepsilon|_{a_n}^{b_n}, \tag{3.14}$$

and $\rho_x^\varepsilon(b_n) = \rho_x^\varepsilon(a_n) = 0$ for $1 \le n \le k-1$ then the only terms that remain from the second term in the right hand side of (3.13) are

$$|\varepsilon H_\delta(\rho^\varepsilon)\rho_x^\varepsilon(a_0)| + |\varepsilon H_\delta(\rho^\varepsilon)\rho_x^\varepsilon(b_k)| \le |\varepsilon H_\delta(\rho^\varepsilon)\rho_x^\varepsilon(0)| + |\varepsilon H_\delta(\rho^\varepsilon)\rho_x^\varepsilon(1)|. \tag{3.15}$$

Thus, by lemma 4, (3.15) is bounded independently of ε, since H_δ is bounded by 1.

So, from (3.13), for each fixed ε, there exists a $\delta_0 = \delta_0(\varepsilon)$ such that

$$(F_{\delta,\varepsilon})_k \le K \quad \text{for } \delta < \delta_0$$

and K independent of ε, δ and k. Let P be any finite partition of I, then there exists a $k > 0$, such that P is contained in a union of sets I_n for $0 \le n \le k$ whose endpoints are extreme points for ρ^ε in I. Now denote $I_k = \bigcup_{n=0}^{k} I_n$ then $I_k \subset I_{k+1}$, and $\lim_{k\to\infty} I_k = I$. Also from (3.12) $(F_{\delta,\varepsilon})_k = \left|\int_{I_k}(\mathcal{H}_\delta(\rho^\varepsilon))_x\right|dx = \int_{I_k}|(\mathcal{H}_\delta(\rho^\varepsilon))_x|dx \le (F_{\delta,\varepsilon})_{k+1}$ which is also bounded independently of δ, ε and k.

Then, by the B. Levi Theorem (see [N])

$$\int_I |(\mathcal{H}_\delta(\rho^\varepsilon))_x|dx = \lim_{k\to\infty}(F_{\delta,\varepsilon})_k \le K$$

for $\delta < \delta_0(\varepsilon)$ and K independent of ε and δ. The proof of lemma 6 is now complete.

Now we are able to prove the following theorem.

Theorem 2. *Let $\{\rho^\varepsilon\}$ be the solution of the "ε-viscosity" problems (2.4), (2.5), and $F(\rho) = j^2\rho^{-1} + \rho^\gamma$. Then, the functions*

$$\mathcal{H}(\rho^\varepsilon) = [F(\rho^\varepsilon) - F(\rho_m)]\,\text{sign}(\rho^\varepsilon - \rho_m) \tag{3.16}$$

are of bounded variation in I and their total variation norm are bounded independently of ε.

Proof. For each fixed ε, we apply Helly's second theorem (see [N], VIII §7) to the family $\{\mathcal{H}_\delta(\rho^\varepsilon)\}_{\delta<\delta_0}$.

Since $\mathcal{H}(\rho^\varepsilon) = \lim_{\delta\to 0}\mathcal{H}_\delta(\rho^\varepsilon)$ uniformly in I, and, from lemma 6, (3.10),

$$TV_I(\mathcal{H}_\delta(\rho^\varepsilon))_{\delta<\delta_0} < K$$

where K is independent of ε and δ, then the total variation of $\mathcal{H}(\rho^\varepsilon)$ in I is

$$TV_I(\mathcal{H}(\rho^\varepsilon)) = \int_0^1 |(\mathcal{H}(\rho^\varepsilon))_x|dx = \lim_{\delta\to o}\int_0^1 |(\mathcal{H}_\delta(\rho^\varepsilon))_x|dx \le K$$

with K independent of ε. □

The next step is to find the limit of the sequence of bounded variation $\mathcal{H}(\rho^\varepsilon)$ functions as ε tends to zero.

Theorem 3. *Under the above conditions, the family $\{\mathcal{H}(\rho^\varepsilon)\}$ has a sequence $\{\mathcal{H}(\rho^{\varepsilon_n})\}$, $\varepsilon_n \to 0$ as $n \to \infty$ that converges pointwise in I and in every $L^p(I)$, $1 \le p < \infty$ to a function $\mathcal{H}_0(x)$ of bounded variation.*

Proof. First, the family of functions $\mathcal{H}(\rho^\varepsilon)(x) = (F(\rho^\varepsilon) - F(\rho_m))\,\mathrm{sign}(\rho^\varepsilon - \rho_m)(x)$ is uniformly bounded in I, as, from (2.11) the family $\{\rho^\varepsilon\}$ is uniformly bounded in I and F is a continuous function of its argument. Also, from theorem 1 we have that the family $\{\mathcal{H}(\rho^\varepsilon)\}$ is uniformly of bounded variation in I and $TV_I(\mathcal{H}(\rho^\varepsilon))$ is bounded by a constant K.

Then an application of the Helly's theorems (see [N] VIII §5–7), assure us there exists a sequence $\{\mathcal{H}(\rho^{\varepsilon_n})\}$, $\varepsilon_n \to 0$ as $n \to \infty$, such that it converges at every point of I to some function $\mathcal{H}_0(x)$ of finite variation, and

$$TV_I(\mathcal{H}_0(x)) \le K.$$

We have to see now that $\{\mathcal{H}(\rho^{\varepsilon_n})\}$ converges to $\mathcal{H}_0(x)$ in $L^p(I)$, $1 \le p < \infty$.

By the Kolmogorov compactness condition theorem (see [N], XVII §3), since the set $\{\mathcal{H}(\rho^\varepsilon)\}$ is uniformly bounded in I, independently of ε (i.e., bounded in $L^p(I)$, $1 \le p \le \infty$), if $\|\mathcal{H}_h(\rho^\varepsilon) - \mathcal{H}(\rho^\varepsilon)\|_{L^p(I)}$ tends zero uniformly in ε, as $h \to 0$, $1 \le p < \infty$, where the subindex h denotes the (Steklov) average function for $\mathcal{H}(\rho^\varepsilon)$ in I, then the family $\{\mathcal{H}(\rho^\varepsilon)\}$ is precompact in L^p, so that the convergence of the sequence $\{\mathcal{H}(\rho^{\varepsilon_n})(x)\}$ to $\mathcal{H}_0(x)$ is in $L^p(I)$.

Indeed, since the family $\{\mathcal{H}(\rho^\varepsilon)\}$ is bounded uniformly in I, independently of ε

$$\|\mathcal{H}_h(\rho^\varepsilon) - \mathcal{H}(\rho^\varepsilon)\|_{L^p(I)} \le C\|\mathcal{H}_h(\rho^\varepsilon) - \mathcal{H}(\rho^\varepsilon)\|_{L^1(I)}.$$

Also, $\mathcal{H}_h(\rho^\varepsilon)(x) = \dfrac{1}{2h}\displaystyle\int_{x-h}^{x+h} \mathcal{H}(\rho^\varepsilon)(\xi)d\xi$ then

$$\begin{aligned}
\mathcal{H}_h(\rho^\varepsilon)(x) - \mathcal{H}(\rho^\varepsilon)(x) &= \frac{1}{2h}\int_{x-h}^{x+h} [\mathcal{H}(\rho^\varepsilon)(\xi) - \mathcal{H}(\rho^\varepsilon)(x)]d\xi \\
&= \frac{1}{2h}\int_{x-h}^{x+h}\left(\int_x^\xi (\mathcal{H}(\rho^\varepsilon))_x(\tau)d\tau\right)(\xi)d\xi = \\
&= \int_0^1 (\mathcal{H}(\rho^\varepsilon))_x(s)\cdot X_{[x-h,x+h]}(s)ds = \int_0^1 (\mathcal{H}(\rho^\varepsilon))_x X_{[-h,h]}(x+s)ds \\
&= \left((\mathcal{H}(\rho^\varepsilon))_x * X_{[-h,h]}\right)(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{H}_h(\rho^\varepsilon) - \mathcal{H}(\rho^\varepsilon)\|_{L^1(I)} &= \|(\mathcal{H}(\rho^\varepsilon))_x * X_{[-h,h]}\|_{L^1(I)} \le \\
&\le \|(\mathcal{H}(\rho^\varepsilon))_x\|_{L^1(I)}\cdot\|X_{[-h,h]}\|_{L^1(I)} = TV_I(\mathcal{H}(\rho^\varepsilon))\cdot 2h \le 2Kh
\end{aligned}$$

with K independent of ε. Hence, the Kolmogorov compactness conditions are satisfied so that the convergence is in $L^p(I)$, $1 \le p < \infty$. The proof of theorem 2 is now complete. □

Next we recall the following result from classical real analysis for which we omit the proof.

Lemma 7. *If $\{g_k\}$ is a uniformly bounded family of $L^1(\Omega)$ functions such that $g_k \to g$ in $L^1(\Omega)$ and σ is a continuous function, then $\sigma(g_k) \to \sigma(g)$ in $L^1(\Omega)$.*

We apply lemma 7 in order to be able to define the weak solution we are searching for. That is, since the function $\mathcal{H}(\rho^\varepsilon(x)) = [F(\rho^\varepsilon) - F(\rho_m)]\,\mathrm{sign}(\rho^\varepsilon - \rho_m)(x)$ is of bounded variation in I and $\mathcal{H}$ is monotone increasing as a function of ρ^ε (since F is a convex function of ρ^ε and $F(\rho_m)$ is the minimum value attained by F), then $\mathcal{H}(x)$ admits an inverse continuous function, denoted by $\mathcal{H}^{-1}(x)$, such that

$$\rho^\varepsilon(x) = \mathcal{H}^{-1}(\mathcal{H}(\rho^\varepsilon)(x)). \tag{3.17}$$

Now, from theorem 2 and lemma 7 we obtain the desired weak solution.

Theorem 4. *For fixed $\varepsilon > 0$, let $\rho^\varepsilon(x)$ denote a solution of (1.12)-(1.16). Then there exists a function $\rho(x)$ that is a weak solution of (1.12)-(1.14), with $\varepsilon = 0$.*

Proof. In view of all previous results, and, from theorem 2 and lemma 7, we obtain that

$$\rho^{\varepsilon_n}(x) \xrightarrow[n\to\infty]{} \mathcal{H}^{-1}(\mathcal{H}_0)(x) \text{ pointwise and in } L^1(I) \tag{3.18}$$

so we define

$$\rho(x) = \mathcal{H}^{-1}(\mathcal{H}_0)(x), \quad x \in I = (0,1). \tag{3.19}$$

Let us see that ρ is a weak solution of the limit equation

$$F(\rho)_x + S(\rho) = 0 \tag{3.20}$$

with $F(\rho) = -j\rho^{-1} + \rho^\gamma$ and $S(\rho) = -\rho w + j\tau^{-1}(\rho, j)$.

From (3.18) and (3.19), we observe that $\rho^{\varepsilon_n} \xrightarrow[n\to\infty]{} \rho$ in $L^1(I)$ and $S(\rho)$ is a continuous function of ρ, then by lemma 7, $S(\rho^{\varepsilon_n}) \xrightarrow[n\to\infty]{} S(\rho)$ in L^1; also, as F is a continuous function of ρ, $F(\rho^{\varepsilon_n}) \xrightarrow[n\to\infty]{} F(\rho)$ in $L^1(I)$.

Now let $\varphi \in C_0^2(I)$ multiply equation (1.13) and integrate by parts, then

$$-\int_I F(\rho^{\varepsilon_n})\varphi_x + \int_I S(\rho^{\varepsilon_n})\varphi + \varepsilon_n \int_I \rho^{\varepsilon_n}\varphi_{xx} = 0. \tag{3.21}$$

Since ρ^{ε_n} are uniformly bounded in I (lemma 3) and φ_{xx} is also bounded in I then the term $\varepsilon_n \int_I \rho^{\varepsilon_n}\varphi_{xx}$ converges to zero as $n \to \infty$. Then taking limit as $n \to \infty$ in (3.21), we obtain that (3.20) is satisfied.

Before going into the "entropy condition" (1.19), we shall observe some easy consequences of the last theorems and lemma 7.

The first one is a trivial observation consequence of theorem 3 and lemma 7.

Lemma 8. *Let $\rho(x)$ be the weak solution of (1.12)-(1.14) for $\varepsilon = 0$, obtained as a limit of ρ^{ε_n} solutions of (1.12)-(1.16), $\varepsilon_n \to 0$, as $n \to \infty$. Then the function defined*

$$\mathcal{H}(\rho)(x) = (F(\rho) - F(\rho_m))\,\mathrm{sign}(\rho - \rho_m)(x) \tag{3.22}$$

is of bounded variation.

Proof. Since $\rho^{\varepsilon_n} \xrightarrow[n\to\infty]{} \rho$ and $F(\rho^{\varepsilon_n}) \xrightarrow[n\to\infty]{} F(\rho)$ both in $L^1(I)$ then $\mathcal{H}(\rho)(x)$ defined as in (3.22) is a continuous function of its argument ρ, and, by lemma 7, $\mathcal{H}(\rho^{\varepsilon_n}) \xrightarrow[n\to\infty]{} \mathcal{H}(\rho)$ in $L^1(I)$.

Now, by Helly's theorem, since $\mathcal{H}(\rho^{\varepsilon_n})$ is uniformly bounded and with uniformly bounded total variation, then $\mathcal{H}(\rho^{\varepsilon_n})(x) \to \mathcal{H}(\rho)(x)$ pointwise in I and $\mathcal{H}(\rho)(x)$ has its total variation bounded.

The next result will give some information about the regularity of the weak solution ρ, when the values of ρ are away from the transonic zone, that is for values of $\rho(x)$ away from ρ_m. The next theorem is an application of the differentiation theorem in $L^1(I)$.

Theorem 5. *Let $\rho(x)$ be the weak solution of (1.12)–(1.14) for $\varepsilon = 0$, obtained as limit of the ε_n-viscosity solutions ρ^{ε_n} of (1.12)–(1.16) as $n \to \infty$. Then, the function $F(\rho)(x) = j\rho^{-1}(x) + \rho^{\gamma}(x)$ is a Lipschitz function in I.*

Proof. Let φ_h be an h-regularization of the characteristic function of the interval $[a, b]$, such that φ_h is an average in the intervals $(a-h, a+h)$ and $(b-h, b+h)$, then if ρ is the weak solution obtained in theorem 3 of problem (1.12)–(1.14) with $\varepsilon = 0$, then from (3.20)

$$\begin{aligned} 0 &= \int_I F(\rho)(\varphi_h)_x + \int_I S(\rho)\varphi_h = \\ &-\left(\int_{a-h}^{a+h} + \int_{b-h}^{b+h}\right) F(\rho)(\varphi_h)_x + \int_{a-h}^{b+h} S(\rho)\varphi_h. \end{aligned} \tag{3.23}$$

Since $0 \le (\varphi_h)_x \le \dfrac{C_1}{h} + C_2$ in $(a-h, a+h)$ and $-\left(\dfrac{C_1}{h} + C_2\right) \le (\varphi_h)_x \le 0$ in $(b-h, b+h)$, for C_1 and C_2 constants, (3.23) implies that

$$\left|\frac{C_1}{h}\int_{a-h}^{a+h} F(\rho) - \frac{C_1}{h}\int_{b-h}^{b+h} F(\rho)\right| \le \mathcal{O}(h)\sup_I |F(\rho)(x)| + \int_{a-h}^{b+h} |S(\rho)\varphi_h|. \tag{3.24}$$

In view of lemma 3, let $B = \sup_I \rho$. Since F is a continuous function of ρ and, by Lemma 7, we have that $F(\rho) \in L^1(I)$, then $\dfrac{1}{h}\displaystyle\int_{a-h}^{a+h} F(\rho)$ converges a.e. to $F(\rho(a))$ as $h \to 0$, and analogously for $F(\rho(b))$.

Taking limit in (3.24) as h goes to zero, since $\sup_I F(\rho) \le F(B)$, we obtain that

$$|F(\rho(a)) - F(\rho(b))| \le (b-a)S(B).$$

Therefore, the function $F(\rho(x))$ is Lipschitz in I.

The result from Theorem 5 can be generalized to any convex function $F(\rho)$ provided the solution ρ^ε of $E_\varepsilon(\rho) = 0$ for the corresponding F are bounded.

An immediate conclusion of Theorem 5 is the following: Since $F(\rho)$ is a decreasing function of ρ for values of ρ less than ρ_m and an increasing function of ρ for values of ρ bigger than m, then in any closed subinterval I' of I such that $\rho(x) < \rho_m$, $x \in I'$ (or $F(\rho(x)) > \rho_m$, $x \in I'$), $F(\rho)(x)$ is a Lipschitz function of x in I', so it admits an inverse defined on $F(I')$. Hence ρ, the weak solution of (1.12)–(1.14) with $\varepsilon = 0$, defined in (3.19), coincides with $F^{-1}(F(\rho(x)))$ in I', which is a Lipschitz continuous in I'. So we set this result as a lemma.

Lemma 9. *The weak solution defined in (3.19) of problem (1.12)–(1.14) for $\varepsilon = 0$ satisfies*

a) *$\rho(x)$ is a Lipschitz continuous function in I', a closed subinterval of I, where $\rho(x) < \rho_m$ for every $x \in I'$. The set of $x \in I'$, such that $\rho(x) < \rho_m$, indicates that I' is inside the supersonic region of equation (3.20).*

b) *$\rho(x)$ is a Lipschitz continuous function in I', a closed subinterval of I, where $\rho(x) > \rho_m$ for every $x \in I'$. The set of $x \in I'$, such that $\rho(x) > \rho_m$ indicates that I' is inside a* subsonic region of equation *(3.20)*.

c) *$\rho(x)$ can have oscillations and discontinuities only at subsets $I' \subset I$ such that ρ takes values above and below ρ_m in subsets of I' with* positive measure. This indicates that *I' is a transonic region of equation (3.20).*

Our final result in section 3 is the derivation of the "entropy condition" stated in (1.19).

Theorem 6 (Entropy Condition). *Let ρ be the weak solution defined in (3.19) of problem (1.12)-(1.14) for $\varepsilon = 0$, then there is a constant $C > 0$ such that the function $\mathcal{H}(\rho)(x)$, defined in (3.22),* satisfies

$$(\mathcal{H}(\rho))_x + C > 0 \tag{3.25}$$

in the sense of the distributions.

Proof. Let $\varphi \in C_0^2(I)$ any positive test function. Multiply (3.5) by φ and integrate , then

$$\begin{aligned} -\int_I \mathcal{H}_\delta(\rho^\varepsilon)\varphi_x + \int_I \mathcal{O}(\delta)\rho_x^\varepsilon\varphi + \int_I (H_\delta S)(\rho^\varepsilon)\varphi = \\ = -\varepsilon \int_I P_\delta(\rho^\varepsilon)\varphi_{xx} + \varepsilon \int_I H_\delta'(\rho^\varepsilon)(\rho_x^\varepsilon)^2\varphi. \end{aligned} \tag{3.26}$$

Let $C = \sup_I S(\rho^\varepsilon)$. $C = C(B)$, where $B = \sup_I \rho^\varepsilon$ for every ε. By lemma 3, B is independent of ε and so is C. Now, any φ regular with compact support in I, satisfies

$$0 = \int_I (Cx\varphi)_x = \int_I Cx\varphi_x + \int_I C\varphi. \tag{3.27}$$

We subtract (3.27) from (3.26) and obtain

$$\begin{aligned} &-\int_I (\mathcal{H}_\delta(\rho^\varepsilon) + Cx)\varphi_x + \int ((H_\delta S)(\rho^\varepsilon) - C)\varphi = \\ &= -\int_I \mathcal{O}(\delta)\rho_x^\varepsilon\varphi - \varepsilon\int_I P_\delta(\rho^\varepsilon)\varphi_{xx} + \varepsilon\int_I H_\delta'(\rho^\varepsilon)(\rho_x^\varepsilon)^2\varphi \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{3.28}$$

First, we look at A_1. Since $|A_1| \le \mathcal{O}(\delta)\varepsilon^{-1}K$ where K depends only on the uniform bound of the family $\{\varepsilon\rho_x^\varepsilon\}$ and the function φ. Then $A_1 \to 0$ uniformly in I as $\delta \to 0$ for each ε fixed.

Second, we look at A_2. Also, since $P_\delta(\rho^\varepsilon) = \int_I H_\delta(\rho^\varepsilon)$, then $P_\delta(\rho^\varepsilon)$ is uniformly bounded in I by a constant independent of ε and δ, so that $A_2 = \mathcal{O}_2(\varepsilon)$ which converges uniformly to zero as ε and δ goes to zero.

Third, $A_3 = \varepsilon\int_I H_\delta'(\rho^\varepsilon)(\rho_x^\varepsilon)^2\varphi \ge 0$, since $H_\delta' \ge 0$ and $\varphi \ge 0$, independently of ε and δ.

Finally, since $(H_\delta S)(\rho^\varepsilon)\varphi \le C\varphi$ in I we have obtained that (3.28) becomes

$$-\int_I (\mathcal{H}_\delta(\rho^\varepsilon) + Cx)\varphi_x = A_1(\delta) + A_2(\varepsilon) + A_3 - \int_I [(H_\delta S)(\rho^\varepsilon) - C]\varphi.$$

Now, taking the limit first as $\delta \to 0$, then as $\varepsilon \to 0$, we obtain that

$$\int_I -(\mathcal{H}(\rho) + Cx)\varphi_x dx \ge 0 \tag{3.29}$$

for any positive test function φ, where ρ is the limit weak solution defined in (3.19).

Hence, (3.29) says that

$$(\mathcal{H}(\rho)+Cx)_x = (\mathcal{H}(\rho))_x + C \geq 0$$

in the sense of the distributions, so that $(\mathcal{H}(\rho))_x$ is a measure bounded below by $-C$. Particularly since $\mathcal{H}(\rho)(x)$ is an $L^1(I)$ function then $\widetilde{G}(x) = \mathcal{H}(\rho(x)) + Cx$ is a monotone increasing function of x in I. Then, $\mathcal{H}(\rho(x))$ is a Lipschitz function plus a monotone increasing function.

Finally, we note that since $F(\rho)$ behaves as a quadratic function of ρ in a neighborhood of ρ_m, then we can say that

$$\rho(x) = \mathcal{H}^{-1}(\mathcal{H}(\rho)) = \mathcal{H}^{-1}(-Cx + \widetilde{G}(x)),$$

where $\widetilde{G}(x)$ is monotone increasing, that is, ρ, is a Hölder-$\frac{1}{2}$ continuous function plus a monotone increasing function with at most a countable set of discontinuities. Also, ρ is not necessarily of bounded variation in a neighborhood of the points x_m such that $\rho(x_m) = \rho_m$ (the neighborhood of such points are transonic regions). Thus, we set this remark as the following lemma which complements Lemma 9.

Lemma 10. *Let ρ be the weak solution defined in (3.19) of problem (1.12)-(1.14) for $\varepsilon = 0$. Then,*

$$\rho(x) = G(x) + \alpha(x)$$

where $G(x) \in C^{1/2}(I)$ and $\alpha(x)$ is monotone increasing in I with at most a countable number of discontinuities.

4. Behavior of the solution near the boundary.

We consider the general form of the ε-viscosity equation and boundary value problem as in (2.5). Thus, let ρ^ε be a solution of

$$E_\varepsilon(\rho^\varepsilon) = 0, \quad \rho^\varepsilon(0) = \rho_0 \text{ and } \rho^\varepsilon(1) = \rho_1 \tag{4.1.a}$$

where

$$E_\varepsilon(\rho) = F(\rho)_x + S(\rho) + \varepsilon\rho_{xx} \tag{4.1.b}$$

where F is the convex function defined in (1.15) and $S(\rho) = S(w, \rho)$ also from (1.15).

In section 2 we proved that the solutions ρ^ε of (4.1) and $\varepsilon\rho^\varepsilon_x$ are uniformly bounded in I independently of ε, and that problem (4.1) is then solvable for every ε.

In section 3 we proved that the function $\mathcal{H}(\rho^\varepsilon)(x) = (F(\rho) - F(\rho_m))\,\text{sign}(\rho - \rho_m)(x)$ has uniform total variation where $F(\rho_m) = \min_{\rho\in(0,\infty)} F(\rho)$ and $\mathcal{H}(\rho^\varepsilon)$ is a monotone function of the variable ρ^ε. Then a weak solution of problem (4.1) for $\varepsilon = 0$ was constructed by taking $\rho(x) = \mathcal{H}^{-1}(\mathcal{H}_0(x))$ where $\mathcal{H}(\rho^\varepsilon(x)) \xrightarrow[\varepsilon\to 0]{} \mathcal{H}_0(x)$ pointwise in $\overline{I}$ and in every $L^p(I)$, $1 \leq p < \infty$ (see theorems 3 and 4).

In lemma 8 we proved that for this weak solution ρ, the function defined by $\mathcal{H}(\rho)(x) = (F(\rho) - F(\rho_m))\,\text{sign}(\rho - \rho_m)(x)$, which is strictly increasing as a function of ρ, has bounded total variation as a function of x, and in theorem 5 we proved that $F(\rho)(x)$ is a Lipschitz function in $\overline{I}$.

The following lemmas and theorems will not depend on the particular form of $F(\rho)$ as long as we have that $F(\rho)$ is strictly convex with a minimum value attained at ρ_m and that we have been able to solve (4.1) for this corresponding $F(\rho)$.

From the regularity obtained for the weak solution ρ (lemmas 9 and 10), ρ has lateral limits at every point of $\overline{I}$, so we state and prove the following lemma.

Lemma 11. *Let ρ be the weak solution of problem (4.1) for $\varepsilon = 0$, defined as in (3.19), and let $\rho_1^- = \lim_{x \to 1^-} \rho(x)$(resp. $\rho_0^+ = \lim_{x \to 0^+} \rho(x)$). Then, for every $\delta > 0$, there exist a σ_δ that depends on δ, such that, for any $\sigma < \sigma_\delta$ fixed, there exist an $\varepsilon_0 = \varepsilon_0(\sigma)$, such that the following holds:*

For each $\varepsilon \leq \varepsilon_0$ there is $x_1^\varepsilon = x_1(\varepsilon) \in \left(1-\sigma, 1-\frac{\sigma}{2}\right)$ $\left(\text{resp. } x_0^\varepsilon = x_0(\varepsilon) \in \left(\frac{\sigma}{2}, \sigma\right)\right)$, where the pair $(\varepsilon, x_1^\varepsilon)$ (resp. $(\varepsilon, x_0^\varepsilon)$) satisfies

$$|\mathcal{H}(\rho^\varepsilon(x_1^\varepsilon)) - \mathcal{H}(\rho_1^-)| < \delta/2 \qquad (\text{resp. } |\mathcal{H}(\rho^\varepsilon(x_0^\varepsilon)) - \mathcal{H}(\rho_0^+)| < \delta/2) \tag{4.2}$$

and either

$$|\rho_x^\varepsilon(x_1^\varepsilon)| \leq \frac{C\delta^{1/2}}{\sigma} \qquad \left(\text{resp. } |\rho_x^\varepsilon(x_0^\varepsilon)| \leq \frac{C\delta^{1/2}}{\sigma}\right) \tag{4.3}$$

or we can choose the sign of ρ_x^ε at x_1^ε(resp. at x_0^ε), i.e.

$$\rho_x^\varepsilon(x_1^\varepsilon) > 0 \quad \text{or} \quad \rho_x^\varepsilon(x_1^\varepsilon) < 0 \qquad (\text{resp. at } x_0^\varepsilon). \tag{4.4}$$

Proof. We shall write the proof for the limit from the left toward the right endpoint of I which is $x = 1$, unless we state otherwise.

Since $\mathcal{H}$ is a Lipschitz function in $\overline{I}$ then $\mathcal{H}(\rho(x)) \to \mathcal{H}(\rho_1^-)$ as $x \to 1^-$, then for a given $\delta > 0$ there exists a $\sigma_\delta = \sigma(\delta)$ such that

$$|\mathcal{H}(\rho(x)) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{8}$$

uniformly in $[1-\sigma, 1]$, for every $\sigma \leq \sigma_\delta$. By theorem 3 in section 2, the family $\{\mathcal{H}(\rho^\varepsilon)\}$ converges pointwise in I to $\mathcal{H}(\rho)$ as $\varepsilon \to 0$, then for ε_1 and ε_2 positive, there exist two points $u \in \left(1-\sigma, 1-\frac{7}{8}\sigma\right)$ and $v \in \left(1-\frac{5}{8}\sigma, 1-\frac{\sigma}{2}\right)$ such that

$$|\mathcal{H}(\rho^\varepsilon(u)) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{4}, \qquad \forall \varepsilon \leq \varepsilon_1 \tag{4.5}$$

and

$$|\mathcal{H}(\rho^\varepsilon(v)) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{4}, \qquad \forall \varepsilon \leq \varepsilon_2. \tag{4.6}$$

Since the solution ρ can be written as $\rho(x) = \mathcal{H}^{-1}(\mathcal{H}(\rho)) = G(x) + \alpha(x)$ where $G(x) \in C^{1/2}(\overline{I})$ and $\alpha(x)$ is monotone increasing (see lemma 10), then, assuming that u and v are points of continuity of ρ,

$$|\rho^\varepsilon(u) - \rho_1^-| + |\rho^\varepsilon(v) - \rho_1^-| < C\delta^{1/2} \qquad \forall \varepsilon \leq \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\},$$

with C independent of ε and δ. Since ρ^ε is a $C^{1,1}(I)$ for every ε, then for any $\varepsilon \leq \varepsilon_0$, by the mean value theorem, there exists an $\widetilde{x}_\varepsilon = \widetilde{x}(\varepsilon)$ and $u \leq \widetilde{x}_\varepsilon \leq v$ such that

$$|\rho_x^\varepsilon(\widetilde{x}_\varepsilon)| \leq \frac{1}{u-v} |\rho^\varepsilon(u) - \rho^\varepsilon(v)| \leq \frac{4}{\sigma} C\delta^{1/2},$$

and C is independent of ε and δ. Now, if the pair $(\varepsilon, \widetilde{x}_\varepsilon)$ satisfies

$$|\mathcal{H}(\rho^\varepsilon(\widetilde{x}_\varepsilon)) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{2}, \tag{4.7}$$

then we choose $x_1^\varepsilon = \widetilde{x}_\varepsilon$ and then condition (4.2) and (4.3) are satisfied.

For those pairs $(\varepsilon, \widetilde{x}_\varepsilon)$ such that (4.7) do not hold, we use the fact that $\mathcal{H}(\rho^\varepsilon(x))$ is a smooth function in $\overline{I}$, for each fixed ε.

Let ε be such that for the pair $(\varepsilon, \widetilde{x}_\varepsilon)$

$$|\mathcal{H}(\rho^\varepsilon(\widetilde{x}_\varepsilon)) - \mathcal{H}(\rho_1^-)| > \delta/2 \tag{4.8}$$

then combining (4.5), (4.6), and (4.8) we can say that

$$\mathcal{H}(\rho^\varepsilon(\widetilde{x}_\varepsilon)) > \{\mathcal{H}(\rho^\varepsilon(v)) + \delta/4\ ,\ \mathcal{H}(\rho^\varepsilon(u)) + \delta/4\} \tag{4.9}$$

or

$$\mathcal{H}(\rho^\varepsilon(\widetilde{x}_\varepsilon)) < \{\mathcal{H}(\rho^\varepsilon(v)) - \delta/4\ ,\ \mathcal{H}(\rho^\varepsilon(u) - \delta/4\} \tag{4.10}$$

with $1 - \sigma \le u \le \widetilde{x}_\varepsilon \le v \le 1 - \dfrac{\sigma}{2}$.

Inequality (4.9) indicates that $\mathcal{H}(\rho^\varepsilon(x))$ has increased from the point u to the point $\widetilde{x}_\varepsilon$ and decreased from the point $\widetilde{x}_\varepsilon$ to v, and (4.10) means the opposite variation. Therefore we can choose points $x^{*,1}$ and $x^{*,2}$ between u and v such that one of the following holds

either a)

$$\mathcal{H}(\rho^\varepsilon(x^{*,1}))_x \ge 0 \text{ and } \mathcal{H}(\rho^\varepsilon(x^{*,1})) - \mathcal{H}(\rho^\varepsilon(u)) < \frac{\delta}{4} \tag{4.11}$$

and

$$\mathcal{H}(\rho^\varepsilon(x^{*,2}))_x \le 0 \text{ and } \mathcal{H}(\rho^\varepsilon(x^{*,2})) - \mathcal{H}(\rho^\varepsilon(v)) < \frac{\delta}{4} \tag{4.12}$$

or b)

$$\mathcal{H}\left(\rho^\varepsilon(x^{*,1})\right)_x \le 0 \text{ and } \mathcal{H}(\rho^\varepsilon(u)) - \mathcal{H}\left(\rho^\varepsilon(x^{*,1})\right) < \frac{\delta}{4} \tag{4.13}$$

and

$$\mathcal{H}\left(\rho^\varepsilon(x^{*,2})\right)_x \ge 0 \text{ and } \mathcal{H}(\rho^\varepsilon(v)) - \mathcal{H}\left(\rho^\varepsilon(x^{*,2})\right) < \frac{\delta}{4} \tag{4.14}$$

with $x^{*,1}$ and $x^{*,2}$ depending on ε.

Since $\mathcal{H}(\rho^\varepsilon)$ is monotone in ρ^ε, then $\mathcal{H}^{-1}$ is also monotone and hence we obtain, combining (4.11)–(4.14) with (4.5)–(4.6) we can find points $x^{*,1}$ and $x^{*,2}$ with $1 - \sigma \le x^{*,1}$, $x^{*,2} \le 1 - \dfrac{\sigma}{2}$ which satisfy

$$|\mathcal{H}(\rho^\varepsilon(x^{*,i}) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{2}, \qquad i = 1, 2$$

and only one of the two following possibilities

$$\rho_x^\varepsilon(x^{*,1}) \ge 0 \text{ and } \rho_x^\varepsilon(x^{*,2}) \le 0 \tag{a}$$

or

$$\rho_x^\varepsilon(x^{*,1}) \le 0 \text{ and } \rho_x^\varepsilon(x^{*,2}) \ge 0. \tag{b}$$

Therefore for the ε's, such that (4.7) is not satisfied. We can choose x_1^ε either $x^{*,1}$ or $x^{*,2}$ such that we can select the sign of the derivative of ρ^ε at the point x_1^ε, and the inequality (4.7) is satisfied, i.e.

$$|\mathcal{H}(\rho_0^\varepsilon(x_1^\varepsilon)) - \mathcal{H}(\rho_1^-)| < \frac{\delta}{2} \tag{4.15}$$

with $1-\sigma \le x_1^\varepsilon \le 1-\frac{\sigma}{2}$ and we can choose either $\rho_x^\varepsilon(x_1^\varepsilon)$ positive or negative, so that (4.2) and (4.4) are satisfied, and thus the proof of the lemma is completed.

Remark. The proof of this lemma can be identically repeated for the left endpoint of I (i.e., $x=0$), in analogy.

Behavior at the right endpoint of I. Let ρ_1 be the value prescribed at the right endpoint of I, i.e. $\rho_1 = \rho^\varepsilon(1)$ for all solutions ρ^ε of problem (4.1).

The use of Lemma 11 will enable us to prove the following theorem.

Theorem 7. *Let us assume that ρ_1 is a subsonic data (i.e. $\rho_1 > \rho_m$) and let $\rho_1^- = \lim_{x\to 1^-} \rho(x)$, where $\rho(x)$ is the weak solution of problem (4.1) for $\varepsilon = 0$, then $\rho_1^- \le \rho_1$. Moreover, if $\rho_1^- < \rho_1$, then ρ_1^- is a supersonic value (i.e. $\rho_1^- < \rho_m$) such that $F(\rho_1) \le F(\rho_1^-)$.*

This last statement indicates that a boundary layer may form for the ε-viscosity solutions.

Proof. We first note that if $\rho_1^- < \rho_1$, then since $\mathcal{H}$ is monotone in ρ, $\mathcal{H}(\rho_1^-) < \mathcal{H}(\rho_1)$. So we let

$$\delta_0 = \frac{1}{4}\left(\mathcal{H}(\rho_1) - \mathcal{H}(\rho_1^-)\right) > 0. \tag{4.16}$$

For any given δ, $0<\delta<\delta_0$ we apply lemma 11 , then we have that there exists a σ_δ, independent of ε, such that for any arbitrary $\sigma < \sigma_\delta$ there exists $\varepsilon_0 = \varepsilon_0(\sigma)$ such that for every $\varepsilon \le \varepsilon_0$, there is a point $x_1^\varepsilon \in \left(1-\sigma, 1-\frac{\sigma}{2}\right)$ which satisfies

$$\left|\mathcal{H}(\rho^\varepsilon(x_1^\varepsilon)) - \mathcal{H}(\rho_1^-)\right| < \delta \tag{4.17}$$

and either

$$|\rho_x^\varepsilon(x_1^\varepsilon)| < \frac{C\delta^{1/2}}{\sigma} \tag{4.18}$$

or we choose

$$\rho_x^\varepsilon(x_1^\varepsilon) \le 0. \tag{4.19}$$

Next, for the same ε of the pair $(\varepsilon, x_1^\varepsilon)$, from (4.6) there is a $v \in \left(1-\frac{5}{8}\sigma, 1-\frac{\sigma}{2}\right)$ such that $\left|\mathcal{H}(\rho^\varepsilon(v)) - \mathcal{H}(\rho_1^-)\right| < \delta/4$, for every $\varepsilon \le \varepsilon_0$. Combining this estimate with (4.16)

$$\mathcal{H}(\rho_1) - \mathcal{H}(\rho^\varepsilon(v)) > 4\delta - \frac{\delta}{4} = C\delta > 0 \text{ for every } \varepsilon \le \varepsilon_0.$$

Since $\mathcal{H}$ is strictly monotone increasing also it is $\mathcal{H}^{-1}$ then this last inequality implies $\rho_1 > \mathcal{H}^{-1}\left(C\delta + \mathcal{H}(\rho^\varepsilon(v))\right)$ then $\rho_1 > \rho^\varepsilon(v)$, $v \in \left(1-\frac{5}{8}\sigma, 1-\frac{\sigma}{2}\right)$, for every $\varepsilon \le \varepsilon_0$. Since $\rho^\varepsilon(x)$ converges to ρ_1 as $x \to 1^-$ then ρ^ε must attain the value ρ_1 at a point in $(v,1]$. Now let $\hat{x}_\varepsilon$ be the first point from the left end of $(v,1]$ such that $\rho^\varepsilon(\hat{x}_\varepsilon) = \rho_1$, then, since $\rho_1 > \rho^\varepsilon(v)$, ρ^ε increases in a neighborhood of $\hat{x}_\varepsilon$. Therefore,

$$\rho_x^\varepsilon(\hat{x}_\varepsilon) \ge 0. \tag{4.20}$$

Next, we denote

$$b_\varepsilon = \hat{x}_\varepsilon \quad \text{and} \quad a_\varepsilon = x_1^\varepsilon \tag{4.21}$$

and we integrate the equation (4.1.b) $E_\varepsilon(\rho^\varepsilon) = 0$ between a_ε and b_ε, and we obtain that

$$F(\rho^\varepsilon(b_\varepsilon)) - F(\rho^\varepsilon(a_\varepsilon)) + (b_\varepsilon - a_\varepsilon)\mathcal{S}(w,\rho^\varepsilon) = \varepsilon(\rho_x^\varepsilon(a_\varepsilon) - \rho_x^\varepsilon(b_\varepsilon)), \tag{4.22}$$

where $\mathcal{S}(w,\rho^\varepsilon) = \dfrac{1}{(b_\varepsilon - a_\varepsilon)}\displaystyle\int_{a_\varepsilon}^{b_\varepsilon} S(w,\rho^\varepsilon)dx$ and $0 \le b_\varepsilon - a_\varepsilon \le 1 - a_\varepsilon \le \sigma$, then $0 \le b_\varepsilon - a_\varepsilon \le \sigma$, and thus, we can estimate the third term of (4.22) as

$$(b_\varepsilon - a_\varepsilon)\mathcal{S}(w,\rho^\varepsilon) \le \sigma\,|\mathcal{S}(w,\rho^\varepsilon)| \tag{4.23}$$

and $S(w,\rho^\varepsilon)$ being smooth and uniformly bounded by $\inf_I w$ and the uniform bound of ρ^ε (see lemma 3), we get $|\mathcal{S}(w,\rho^\varepsilon)|$ uniformly bounded by a constant K independently on ε.

Combining (4.18), (4.19), (4.20), and (4.21) we get that $\rho_x^\varepsilon(b_\varepsilon) \ge 0$ and $|\rho_x^\varepsilon(a_\varepsilon)| \le \dfrac{C\delta^{1/2}}{\sigma}$ or $\rho_x^\varepsilon(a_\varepsilon) \le 0$. Also, we have that $F(\rho^\varepsilon(b_\varepsilon)) = F(\rho_1)$.

Therefore, replacing in (4.22)

$$\begin{aligned} F(\rho_1) - F(\rho^\varepsilon(a_\varepsilon)) &\le \sigma K + \varepsilon\,[\rho_x^\varepsilon(a_\varepsilon) - \rho_x^\varepsilon(b_\varepsilon)] \le \\ &\le \sigma K + \varepsilon\frac{C\delta^{1/2}}{\sigma} \qquad \forall\varepsilon \le \varepsilon_0. \end{aligned} \tag{4.24}$$

Finally,

$$F(\rho_1) - F(\rho_1^-) \le \sigma K + \varepsilon\frac{C\delta^{1/2}}{\sigma} + F(\rho^\varepsilon(a_\varepsilon)) - F(\rho_1^-)$$

and from (4.17) $|F(\rho^\varepsilon(a_\varepsilon)) - F(\rho_1^-)|$, which is bounded by $|\mathcal{H}(\rho^\varepsilon(a_\varepsilon)) - \mathcal{H}(\rho_1^-)|$, is less than δ.

Hence,

$$F(\rho_1) - F(\rho_1^-) \le \sigma K + \varepsilon\frac{C\delta^{1/2}}{\sigma} + \delta \qquad \forall\varepsilon \le \varepsilon_0(\sigma),\ \sigma \le \sigma_\delta. \tag{4.25}$$

where K and C are independent of ε , δ and σ, and δ and σ are arbitrary. Letting first ε go to zero, we obtain

$$F(\rho_1) - F(\rho_1^-) \le \sigma K + \delta,$$

where σ and δ are arbitrary. Therefore

$$F(\rho_1) \le F(\rho_1^-). \tag{4.26}$$

Since F is strictly monotone increasing for values of $\rho \ge \rho_m$, then since we assume $\rho_1^- < \rho_1$ and $\rho_1 > \rho_m$ then ρ_1^- must be less than or equal to ρ_m. It remains to prove that ρ_1^- cannot be bigger than ρ_1. In order to obtain this result we use Lemma 11 in the following form: Assume $\rho_1^- > \rho_1$, and again we let $\delta_0 = \frac{1}{4}\mathcal{H}(\rho_1^-) - \mathcal{H}(\rho_1) > 0$. Then, as we did in i) for any $0 < \delta \le \delta_0$ and , we find there is a σ_δ, such that for every arbitrary $\sigma < \sigma_\delta$, there exist $\varepsilon_0 = \varepsilon_0(\sigma)$, such that for every $\varepsilon \le \varepsilon_0$ there is a point x_1^ε in $\left(1-\sigma, 1-\frac{\sigma}{2}\right)$ which satisfies

$$\left|\mathcal{H}(\rho^\varepsilon(x_1^\varepsilon)) - \mathcal{H}(\rho_1^-)\right| < \delta \tag{4.27}$$

and either

$$|\rho_x^\varepsilon(x_1^\varepsilon)| < \frac{C\delta^{1/2}}{\sigma}$$

or we choose

$$\rho_x^\varepsilon(x_1^\varepsilon) \geq 0.$$

With a similar argument we can now find a $\hat{x}_\varepsilon$ in $(v, 1]$ such that $\rho^\varepsilon(\hat{x}_\varepsilon) = \rho^\varepsilon(1) = \rho_1$ but with the property that $\rho_x^\varepsilon(\hat{x}_\varepsilon) \leq 0$ (this property is valid because we have assumed $\delta_0 > 0$). Then again choosing $b_\varepsilon = \hat{x}_\varepsilon$ and $a_\varepsilon = x_1^\varepsilon$ and repeating the computation from (4.22) and (4.24) we get

$$\begin{aligned} F(\rho_1) - F(\rho_1^-) &> -\sigma K + \varepsilon\left[\rho_x^\varepsilon(a_\varepsilon) - \rho_x^\varepsilon(b_\varepsilon)\right] \\ &\geq -\sigma K - \varepsilon\frac{C\delta^{1/2}}{\sigma} + F(\rho^\varepsilon(a_\varepsilon)) - F(\rho_1^-). \end{aligned}$$

By (4.26),

$$F(\rho_1) - F(\rho_1^-) \geq -\sigma K - \varepsilon\frac{C\delta^{1/2}}{\sigma} - \delta$$

for every $\varepsilon \leq \varepsilon_0(\sigma)$, with K, C, independent of ε,δ and σ. First, letting ε go to zero, since σ and δ are arbitrary, then we get

$$F(\rho_1) \geq F(\rho_1^-). \tag{4.28}$$

Therefore, if $\rho_1^- > \rho_1$ with $\rho_1 \geq \rho_m$, since F is strictly monotone increasing in (ρ_m, ∞), then $F(\rho_1) < F(\rho_1^-)$ which contradicts (4.27). This concludes that $\lim_{x\to 1^-} \rho(x) \leq \rho_1$, and if $\lim_{x\to 1^-} \rho(x) < \rho_1$ then

$$F(\rho_1) \leq F\left(\lim_{x\to 1^-} \rho(x)\right) \text{ and } \lim_{x\to 1^-} \rho(x) \leq \rho_m.$$

Theorem 7 has been proved.

Behavior at the left endpoint of I.

Theorem 8. *Let us assume that ρ_0 is a subsonic data (i.e. $\rho_0 > \rho_m$) and let $\rho_0^+ = \lim_{x\to 0^+} \rho(x)$, where $\rho(x)$ is the weak solution of problem (4.1) for $\varepsilon = 0$, then $\rho_0^+ \geq \rho_m$.*

Proof. We again use Lemma 11 for the corresponding form at the left endpoint of I, $x = 0$. This proof is very similar to the proof of the first part of Theorem 7. However, we shall write it again.

Let us assume $\rho_0^+ < \rho_m$, then we set

$$\delta_0 = \frac{1}{8}\left(\mathcal{H}(\rho_m) - \mathcal{H}(\rho_0^+)\right) = -\frac{\mathcal{H}}{8}(\rho_0^+) > 0. \tag{4.29}$$

Then for every $0 < \delta < \delta_0$, there exists a σ_δ such that for every $\sigma < \sigma_\delta$, there exist an $\varepsilon_0 = \varepsilon_0(\sigma)$ such that for each $\varepsilon \leq \varepsilon_0$ there is a point $x_0^\varepsilon \in \left(\frac{\sigma}{2}, \sigma\right)$ which satisfies

$$|\mathcal{H}(\rho^\varepsilon(x_0^\varepsilon) - \mathcal{H}(\rho_0^+)| < \delta$$

and either

$$|\rho_x^\varepsilon(x_0^\varepsilon)| < \frac{C}{\sigma}\delta^{1/2}$$

or we choose

$$\rho_x^\varepsilon(x_0^\varepsilon) \geq 0.$$

Next, for the same ε of the pair $(\varepsilon, x_0^\varepsilon)$, from (4.5), there is a $u \in \left(\frac{\sigma}{2}, \frac{5}{8}\sigma\right)$ such that $|\mathcal{H}(\rho^\varepsilon(u)) - \mathcal{H}(\rho_0^+)| < \frac{\delta}{4}$ for every $\varepsilon \leq \varepsilon_0$. Then, combining this estimate with (4.29) we have that $-\mathcal{H}(\rho^\varepsilon(u)) > -\frac{\delta}{4} + 8\delta = c\delta > 0$. Since $\mathcal{H}$ is strictly monotone increasing, it is also $\mathcal{H}^{-1}$. The last inequality implies $(\rho_m > \mathcal{H}^{-1}(c\delta + \mathcal{H}(\rho^\varepsilon(u)))$ then

$$\rho_m > \rho^\varepsilon(u), \quad u \in \left(\frac{\sigma}{2}, \frac{5\sigma}{8}\right), \text{for every } \varepsilon \leq \varepsilon_0. \tag{4.30}$$

Since $\rho^\varepsilon(x)$ converges to ρ_0 as $x \to 0^-$ and $\rho_0 > \rho_m$, then ρ^ε must attain the value ρ_m at a point in $(0, u)$. Now let $\hat{x}_\varepsilon$ be the first point from the right end of $(0, u)$ such that $\rho^\varepsilon(\hat{x}_\varepsilon) = \rho_m$, so that by (4.30) ρ^ε decreases in a neighborhood of $\hat{x}_\varepsilon$, and then $\rho_x^\varepsilon(\hat{x}_\varepsilon) \leq 0$.

Next we denote

$$b_\varepsilon = x_0^\varepsilon \quad \text{and} \quad a_\varepsilon = \hat{x}_\varepsilon$$

and we integrate the equation (4.1) $E_\varepsilon(\rho^\varepsilon) = 0$ between a_ε and b_ε. We obtain a similar expression to (4.22)

$$F(\rho^\varepsilon(b_\varepsilon)) - F(\rho^\varepsilon(a_\varepsilon)) + (b_\varepsilon - a_\varepsilon) S(w, \rho^\varepsilon) = \varepsilon(\rho_x^\varepsilon(a_\varepsilon) - \rho_x^\varepsilon(b_\varepsilon))$$

where $|S(w, \rho^\varepsilon)| \leq K$ and $0 \leq b_\varepsilon - a_\varepsilon \leq u - a_\varepsilon \leq \sigma$, and $F(\rho^\varepsilon(a_\varepsilon)) = F(\rho_m)$ and $\rho_x^\varepsilon(a_\varepsilon) \leq 0$.

Since $\rho_x^\varepsilon(b_\varepsilon) \geq 0$ or $\rho_x^\varepsilon(b_\varepsilon) \leq \frac{C}{\sigma}\delta^{1/2}$ we get that

$$F(\rho_0^+) - F(\rho_m) \leq \sigma K + \varepsilon\frac{C}{\sigma}\delta^{1/2} + |F(\rho_0^+) - F(\rho^\varepsilon(b_\varepsilon))|.$$

By (4.12) of lemma 11 (corresponding at the left side) we have that

$$\left|F(\rho_0^+) - F(\rho^\varepsilon(b_\varepsilon))\right| = \left|\mathcal{H}(\rho_0^+) - \mathcal{H}(\rho^\varepsilon(x_0^\varepsilon))\right| < \delta$$

so that

$$F(\rho_0^+) - F(\rho_m) \leq \sigma K + \varepsilon\frac{C}{\sigma}\delta^{1/2} + \delta$$

for every $\varepsilon \leq \varepsilon_0(\sigma)$ with K and C independent of ε , δ and σ, and, δ and σ arbitrary. Therefore, taking first limit as ε goes to zero,

$$F(\rho_0^+) \leq F(\rho_m)$$

which contradicts the fact that $\rho_0^+ < \rho_m$ and that F is strictly decreasing in $(0, \rho_m)$ and $F(\rho_m) = \min_{(0,\infty)} F(\rho)$. Theorem 8 has been proved.

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